DEVELOPMENT OF APPLICATION OF PLASTIC HINGES
TO REINFORCED CONCRETE THEORY

by

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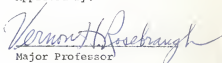
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SYNOPSIS

This report presents a review of the development of limit design of reinforced concrete structures. A brief comparison of plastic design of steel and limit design of structural concrete is presented to introduce the different problems that must be considered in each design procedure. Experimental evidence of the inelastic behavior of reinforced concrete is discussed, with emphasis placed on moment redistribution and the rotational capacity of hinging regions. Various theories of limit design, based on experimental results and further analytical work, are introduced.

A brief discussion of foreign design codes is included to illustrate the practical use of limit design, and the importance of incorporating limit design into United States building codes is stressed.

INTRODUCTION

Inelastic behavior of reinforced concrete has played an important part in American and foreign design recommendations. In America the first step was the introduction of ultimate strength design in the 1956 revision of the ACI Building Code. The second step was the introduction of limit design for statically indeterminate concrete structures which allows readjustment in the relative magnitudes of internal forces and moments at sections subjected to high loads.

Many countries base their codes of practice on the fact that the non-linear relationships between stress and strain for concrete and steel affect the ultimate strength of individual cross sections as well as the ultimate strength of an indeterminate structure as a whole. These countries have all taken slightly different approaches to an eventual full consideration of plasticity in design practice. Denmark and Norway carry out moment redistribution by inelastic action, but the design of sections is based on straight line theory and allowable working stress. In Russia and England non-linear relationships are used in both determining the distribution of design moments and determining the ultimate strength of various sections.

Limit design does not mean that the final design of cross-sections be done on an inelastic basis. Ultimate strength design or straight-line theory may be used to

design various sections. Therefore, it appears that the logical approach would be to use inelastic methods of design for both individual sections and the whole structure by combining limit design and ultimate strength design. This cannot be done until limit design is introduced into the A.C.I. Building Code. At present, a Joint A.C.I.-A.S.C.E. Committee on Limit Design (formed in 1957) is guiding the development of structural concrete design involving limit design, and many American structural concrete designers are anxiously waiting for the results of this committee.

TERMINOLOGY

Redistribution of Moments - results in a statically indeterminate structure by the formation of plastic hinges until the ultimate load is reached. As a result of the formation of plastic hinges, less-highly stressed portions of a structure may carry increased moments.

Yield Moment - in a member subject to bending, is a moment at which an outer fiber first attains yield point stress.

Plastification - is the general penetration of yield stress from an outer film toward the centroid of a section under increasing moment. Plastification is complete when the plastic moment is attained.

Ultimate Moment or Plastic Moment - is the maximum moment of resistance of a fully yielded cross section.

Plastic Hinge - is a yielded zone which forms in a structural member when the plastic moment is applied. The plastic hinge is capable of rotation as if the member were hinged, but the plastic moment is maintained at the hinge.

Hinge Angle or Hinge Rotation - is the angle through which the considered plastic hinge must rotate under the plastic moment before a sufficient number of further hinges form to develop a mechanism.

Rotation Capacity - is the angular rotation which a given member can sustain at the plastic moment without local failure at the plastic hinge.

Mechanism – is an articulated system of structural members connected by plastic and/or real hinges. The system is able to deform without finite increase in load until the deformations become so large that the equilibrium equations no longer apply.

Ultimate Load or Plastic Load Limit – is the load attained when a sufficient member of plastic hinges have formed to transform either the whole or part of a structure into a mechanism. It is the largest load a structure can be counted upon to support.

Working Loads – are the loads the structure is expected to carry, permanently or temporarily, during its useful life.

Load Factors – are factors by which the service loads are multiplied to obtain the design ultimate loads. This choice of terms serves to emphasize the reliance upon load-carrying capacity of the structure rather than upon stress.

Straight Line Theory – is a method of structural concrete design assuming a straight-line relationship between stress and strain of concrete and reinforcement, with some empirical modifications and adjustments to take into account certain inelastic characteristics observed in tests.

Ultimate Strength – is the largest moment, axial force, or shear a structural concrete cross section will support.

Ultimate Strength Design – is a method of design based on ultimate strength by inelastic action of structural concrete cross sections subject to bending, axial force, shear, bond,

or combinations thereof. Ultimate strength design does not
necessarily involve an inelastic theory of statically indeter-
minate structures. Evaluation of external moments and forces
that act in indeterminate structures by virtue of dead and
live loads may be carried out by the theory of elastic dis-
placements, or by limit design and yield line theory.

Limit Design - is an inelastic theory of statically
indeterminate concrete structures in which readjustments in
the relative magnitude of internal moments and forces at
various sections are recognized at high loads. Limit design
does not necessarily involve a final design of cross sections
on an inelastic basis. Sections may be designed by ultimate
strength design, or by straight line theory.

Yield Line Theory - is a branch of Limit Design. It is
a theory of reinforced concrete slab design based on inelastic
action occurring in a pattern of yield lines, the location
of which depends on loading and boundary conditions. Final
design of cross sections may be carried out by ultimate
strength design or straight line theory.

Plastic Design - is a structural steel design method
which defines the limit of structural usefulness as the load
attained when a sufficient number of plastic hinges have
formed to transform the structure into a mechanism.

Plastic Modulus - is the bending resistance modulus of a
fully yielded cross section. It is the combined statical
moments about the nutral axis of the cross sectional areas
above and below that axis.

Shape Factor - is the ratio of the plastic moment to the yield moment for a given cross section.

Working Stress Design - is a structural steel design method which defines the structural limit at the load at which a calculated stress equal to the yield point is first attained at any point. Local stress raisers are generally disregarded.

Factor of Safety - as used in working stress design is a factor by which the yield stress of the material is divided to obtain a workable or allowable stress.

CONCEPT OF PLASTIC DESIGN

An advantageous replacement for conventional elastic
design of statically loaded structural frames and beams is
the plastic design concept. Elastic design theory limits the
maximum load which a structure can support to the load which
first causes a stress somewhere in the structure to equal the
yield point of the structural material. Designers are well
aware that ductile materials (steel) will continue to yield
without failure long after the yield stress has been reached.
When the stress at one point in a ductile structure reaches the
yield stress of the material, a zone of local yielding will
develop. As the load on the structure is increased, the
stress at the zone of local yielding will remain approximately
constant, and the less stressed parts of the structure will be
required to take the increased load. (Because the structure
is still elastic outside of the yielded zone, the total deflec-
tion of the structure is controlled.) As the loading continues
to increase other yield zones will develop, and when a sufficient
number of these yield zones have developed, the structure will
fail. It is suggested that plastic design be applied to indeter-
minate beams and frames only. Statically determinate structures
can resist little load in excess of the amount that causes
first development of yield stress.

There are many reasons why plastic design should be
considered in frames and beams (steel and reinforced concrete).

The following are but a few of the advantages of plastic design:

1) A considerable saving in steel (10 or 15 percent for some structures) can be accomplished.

2) The designer can make a more accurate estimate of the actual collapse strength of the structure and the maximum allowable load.

3. For many structures plastic analysis is easier to apply than elastic analysis.

4. By permitting plastic deformation a structure can be subjected to large unpredictable stresses.

Figures 1 and 2 further illustrate the advantage of plastic design as compared with elastic design.
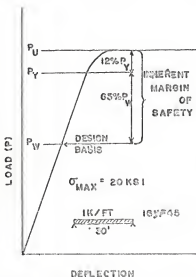

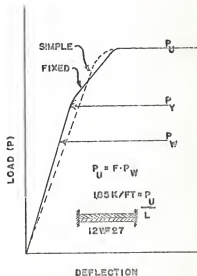FIG. 1. LOAD DEFLECTION CURVE FOR ELASTIC DESIGN

FIG. 2. LOAD DEFLECTION CURVE FOR PLASTIC DESIGN

The moment diagram for the beam in Fig. 1 has only one point of maximum moment, and the ultimate load carrying capacity is only slightly greater than Py. In Fig. 2 the moment diagram has three peaks, and the ultimate load is not reached until yield zones have formed at the ends and in the center. When the ends reach the yield stress, one can see from the figure that there is still a lot of load carrying capacity before the ultimate load of the beam is reached. Therefore, the purpose of plastic design is to relate the working load to the ultimate load thus found.

The application of plastic analysis to structural design was introduced by Dr. Galior Kazinezy, a Hungarian, who published results of tests on clamped girders as early as 1914. The American Institute of Steel Construction, the Welding Research Council, the Navy Department, and the Iron and Steel Institute have sponsored studies of the plastic behavior of steel structures at Lehigh University. These studies have served to verify the plastic method of analysis, and have made it possible to design numerous structures in both Europe and North America based upon this method. Plastic design of steel frames has advanced to the point where it is now a part of various building and steel specifications.

With the general concepts of plastic design of steel frames accepted by various building codes around the world, one wonders how long it will be before limit design (the second logical step in the recognition of the inelastic behavior of reinforced

concrete structures) will be recognized as an acceptable design procedure in America for reinforced and prestressed concrete structures.

LIMIT DESIGN OF STRUCTURAL CONCRETE

There are two very important differences between the plastic design of structural steel and the limit design of structural concrete. These differences are the distribution of moment resistance and rotation capacity.

Distribution of Moment Resistance - Plastic design of structural steel allows considerable savings in cost when compared to elastic design, because the positive and negative moment resistance of structural steel members are constant and equal along the length of a member, unless the depth varies or coverplates are used. Plastic design must also satisfy the conditions of equilibrium, collapse, and yield.

Because the amount and location of steel in a reinforced concrete member can be varied, it is possible to vary the positive and negative resisting moment as well as the moment capacity along the length of the member. Therefore, it is possible to reinforce a structure so that the distribution of moments at ultimate load is very close to the moments found by elastic moment distribution. If this procedure is followed, all the necessary plastic hinges will form at the same time, and very little hinge rotation is required. This design procedure may also be reversed. Arbitrary locations and plastic moments can be chosen to satisfy equilibrium conditions for the chosen mechanism. Then the reinforcement can be proportioned to avoid yielding between the chosen hinge locations.

Rotation Capacity - The plastic design of steel struc-
tures is based on forming enough plastic hinges to change all
or part of the structure into a mechanism causing collapse.
Little consideration is given to how much one hinge section
is strained, because the ultimate strain of mild steel is
greater than 15 per cent. This 15 per cent is much greater
than any strain obtainable by moment distribution in any one
section.

For concrete in flexural compression the ultimate strain
is from 0.3 to 0.5 per cent, while the ultimate strain for
tension reinforcement varies from 0.5 to 2.0 per cent. There-
fore, the rotational capacity for a reinforced concrete section
must be more carefully considered than for structural steel.
To avoid excessive flexural cracking the rotation of a structural
concrete section has to be limited, even though the section
still possesses additional rotation capacity.

It is in the preceding two respects that plastic design
of structural steel differs from limit design of structural
concrete. Further examination of the distribution of moment
resistance and the rotation capacity of reinforced concrete
is presented in this paper.

EXPERIMENTAL RESULTS OF MOMENT REDISTRIBUTION IN
REINFORCED CONCRETE

In 1920 the German Reinforced Concrete Committee[1] con-
ducted the first experimental work to demonstrate moment
redistribution in reinforced concrete beams. They tested
two beams fully fixed at the ends, and the results showed
that for strength design purposes, the magnitudes of the
end fixing moments relative to the span moment could be
arbitrarily chosen, as long as static equilibrium was main-
tained.

In 1930 Von Emperger[2] tested moment redistribution of
reinforced concrete beams, and recommended that one can
assume an arbitrary moment diagram in equilibrium with the
external loads if designing a beam with unknown fixity.

The first extensive tests demonstrating moment redis-
tribution in reinforced concrete beams were conducted by
G. von Kazinezy[3], who was the first to introduce the
development of plastic hinges in structural steel. In
1933, he tested ten two-span continuous beams loaded at
their third-points. The beams were incorrectly over-reinforced
in span sections and some in support sections (Fig.3). This
means that the beam had a stronger section than that required
by elastic theory. The tests showed that all beams failed
when both the span and support sections reached their maximum
moment capacity as calculated by the ultimate strength theory
of that period. Redistribution of moments was a result of the

ductile behavior of the reinforcing steel. In these early investigations, plastic rotation and ultimate strength of all sections were governed by yielding of ductile reinforcing steel.
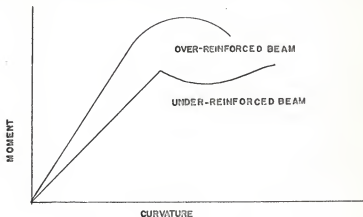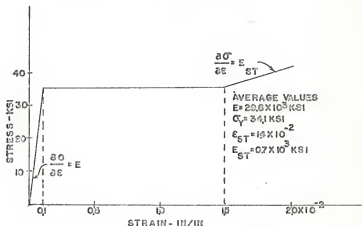

FIG.3. TYPICAL MOMENT-CURVATURE CURVE

FIG 4 STRESS-STRAIN CURVE OF A7 STRUCTURAL STEEL

The ductility of steel is evident from Fig. 4 which shows in somewhat idealized form the stress-strain properties of steel in the initial portion of the curve.

Further investigation of the redistribution of bending moments in reinforced concrete beams and frames as a result of inelastic behavior, both of the steel and of the concrete, was conducted by W. H. Glanville and F. G. Thomas.[4] The test specimens were two-span continuous beams loaded with a concentrated load in each span. The center support sections failed by yielding of the steel in two beams and by crushing of the concrete in six. Two more beams had compression reinforcement at the center support, but in such a manner so as to fail in compression. In all the tests except the two with compression reinforcement, it was found that moment redistribution continued until both support and span sections reached their ultimate resisting moment. The two beams with compression reinforcement failed at the support before the span section had developed its maximum strength.

Four pin-ended single-bay portal frames were also tested. Primary failure by yielding of steel and primary failure by crushing of concrete in the columns were considered. Full redistribution occurred in the first case, while in the latter case the columns failed first. In all of the preceding tests only one percentage of reinforcement was considered. Therefore, the results of those tests were not conclusive, although they do verify that moment redistribution does occur in reinforced concrete continuous structures. Further work was therefore

necessary in order to predict the safe degree of redistribution in any one particular structure.

Professor A. L. L. Baker[5] carried on a continuous investigation of the problems of limit design at the Imperial College, University of London. Many tests were carried out to provide information as to the strength and rotational capacity of hinging sections. One reinforced concrete frame was tested in a manner to simulate vertical and horizontal loading of a part of a building frame. The frame was designed so that hinging sections would occur in the compression members. Figure 5 shows the frame tested, and that the last measured moment ratio before failure is close to the ultimate strength of the critical sections.
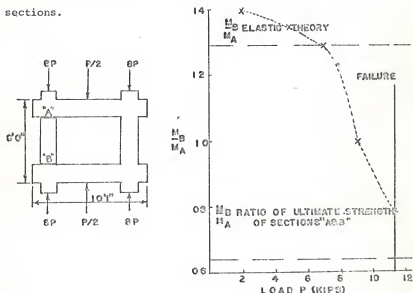
FIG.5. REINFORCED CONCRETE FRAME TESTED & RESULTS

The influence of the percentage of lateral ties on the maximum concrete strain at ultimate strength was also investigated. The specimens used were subjected to various combinations of axial load and bending moments. It was concluded that the closer the lateral ties were spaced, the higher the maximum concrete stain at ultimate strength. The results of one such test can be seen in Fig. 6.
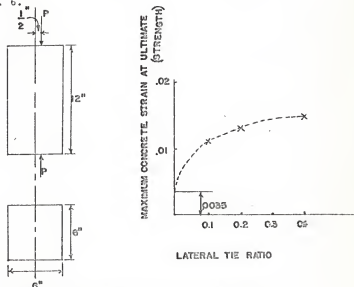
4 NO.5 BARS WITH VARYING AMOUNTS OF LATERAL TIES

FIG.6. ECCENTRICALLY LOADED CONCRETE PRISMS

A prestressed concrete portal frame was tested to destruction, and it was found that the bending moment in the beam was 74 per cent of the corner moment during the working load range (elastic behavior). The distribution of bending moments at failure was controlled by the ultimate strengths of the critical

sections, and the moment in the beam was 168 per cent of the corner moment. Professor A. L. L. Baker used the results of his many tests to develop a limit design theory which will be discussed later.

The Concrete and Cement Association[6] of Europe has carried out many tests on the redistribution of moments in prestressed and reinforced concrete beams and frames. The positive results of these tests have determined the extent in which limit design has become an accepted part of European design practice.
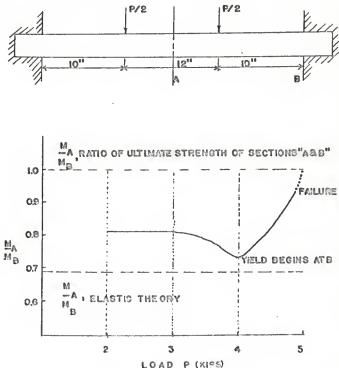
FIG 7   REINFORCED CONCRETE BEAM WITH FIXED ENDS

In 1953, Professor L. H. N. Lee[7] tested a group of fixed-end reinforced concrete beams, to examine the redistribution of bending moments at high loads. Figure 7 shows the results of one of those tests in which the ultimate strengths of sections A and B were made equal. Measurements began after the initial cracking of section B, but before the cracking of section A. Redistribution occurred during this time due to the stiffness variation along the beam. Section A started to crack at a load of 3 kips and distribution approached that predicted by elastic theory. At a load of 4 kips section B started to yield and redistribution of moments occurred until full redistribution was reached at failure.

STUDY OF ROTATIONAL CAPACITY OF REINFORCED CONCRETE

ROTATIONAL CAPACITY OF HINGING REGIONS
IN REINFORCED CONCRETE BEAMS

As previously stated, one major difference between limit

design of structural concrete and plastic design of structural

steel is that little attention is given to the magnitude of

strains at the individual hinging sections as redistribution

of moment occurs in plastic design. The strains that can be

developed in a mild steel member are considerably greater than

those than can be developed in a reinforced concrete member.

It is possible that the strain capacity of a reinforced con-

crete hinging section can reach its limit before full

redistribution of moments has taken place. Therefore, it

is necessary to give careful consideration to hinging regions,

and to limit their rotation to known safe values. An inves-

tigation of this problem was performed by Alan H. Mattock[89] to

study the moment-rotation characteristics of reinforced con-

crete beams in the region of a peak in the moment diagram,

such as occurs at a support in a continuous beam. Most of the

test specimens were simple-span beams subjected to a concen-

trated load at midspan, in order to simulate the distribution

of moments adjacent to a support in a continuous beam (Fig. 8).

Thirty-seven beams were tested involving the following variables:

strength of concrete, depth of beam, distance from point of

maximum moment to a point of zero moment, and amount and yield point of reinforcement (Fig. 9 and 10). The framework for this investigation was laid to study the influence of several variables on the moment-curvature relationships for reinforced concrete sections adjacent to a support, and on the total rotational capacity of the hinging regions adjacent to a support.

An interesting aspect of the experimental operation was the instrumentation used. In order to obtain true values of rotation, deflection, and strain at ultimate strength, electrical measuring devices coupled to a set of Sanborn type 67A continuously-recording units were used. Strains increase rapidly as ultimate strength is approached. If incremental readings had been taken, failure may have occurred while the load was being increased from one stage to the next. If this happened, the last set of measurements would correspond to the load stage before failure, and not to the instant at which ultimate strength was attained.

Much success was obtained in the measurement of rotation in the midspan region using differential transformer gages, and in the estimation of total rotation from the measured midspan deflections. These results have enabled a considerable insight to be obtained into the rotational capacity of hinging regions in reinforced concrete beams.

All the beams were under reinforced and failed by crushing of the concrete after the tension reinforcement had yielded by varying degrees. Flexural cracks first occurred at between 15 and 30 percent of ultimate load, and these cracks widened and
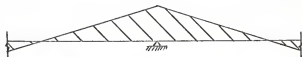
BENDING MOMENT DISTRIBUTION NEAR A CONTINUOUS BEAM SUPPORT
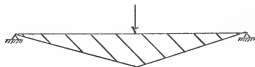
BENDING MOMENT DISTRIBUTION IN A TEST BEAM

FIG.8. RELATIONSHIP BETWEEN DISTRIBUTION OF MOMENTS IN THE TEST BEAMS AND THOSE NEAR A SUPPORT IN A CONTINUOUS BEAM.

NO 3 BARS
NO 2 BARS ON 2.5,5, OR 10 IN. CENTERS

NO 6 BARS

10"
OR
20"

6"

5' 7", 10' 2", 19' 4"
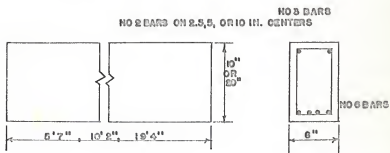
FIG.9. TYPICAL DETAILS OF TEST BEAMS
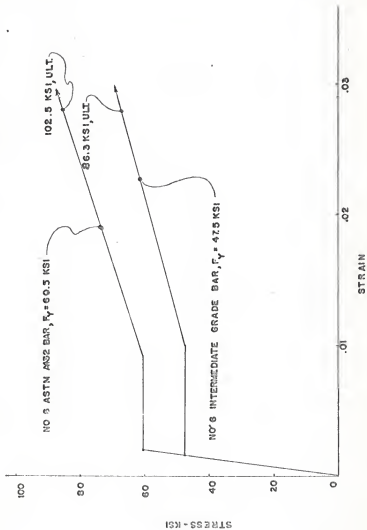
FIG.10. TYPICAL STRESS-STRAIN CURVES FOR TENSION STEEL

102.5 KSI, ULT.

86.3 KSI, ULT.

NO 8 ASTM A432 BAR, F_y = 60.5 KSI

NO 6 INTERMEDIATE GRADE BAR, F_y = 47.5 KSI

extended toward the compression zone as the load increased.
The cracks widened rapidly and increased in number at the
level of the tension reinforcement after yield of the tension
reinforcement. In most cases failure of the concrete com-
pression zone did not occur until a maximum concrete compression
strain in excess of .003 was reached. Generally the ultimate
moment was in excess of the yield moment depending upon the
degree of reinforcement and the length of the span.

From the linear differential transformer gage readings
at midspan, moment-curvature and moment-deflection diagrams
were obtained. A study of these data for the various beams
tested led to the following qualitative conclusions.

√ For a given span, the curvature at ultimate strength
decreases as the tension reinforcement index increases. There-
fore, the curvature is inversely proportional to the neutral
axis depth, which at ultimate strength varies directly as the
tension reinforcement index if the compression reinforcement
index is constant.

$$(1) \quad \text{Curvature, } \gamma = \frac{\text{Maximum concrete compressive strain}}{\text{Neutral axis depth}}$$

√For a given reinforcement index and depth of beam, the
ultimate curvature at midspan decreases as the distance, z,
from point of maximum moment to point of zero moment increases,
and tends toward the value of ultimate curvature measured in
the region of constant moment in a beam loaded at two points.

A plot of the maximum concrete compressive strain at mid-
span against the reciprocal of the distance from the point of

maximum moment to the point of zero moment, $\frac{1}{6}$, results in
a straight line graph with the slope equal to the ultimate
strain.   (2) $\varepsilon_u = 0.003 + \frac{0.5}{6}$

In a region of constant moment this equation yields a value
of .003, which is the value used for ultimate strength cal-
culations in the ACI Building Code. The preceding equation
is not dimensionally correct, and because of the variable in
this test procedure, it is subject to further refinement when
additional test results become available.

The inelastic deformations are not all concentrated at
the section of maximum moment, and plasticity may extend
beyond the distance d/2 from the section depending on the
ratio of the distance from the point of maximum moment to the
point of zero moment, θ to the effective depth, d, of the
section, and on the degree of flexural reinforcement of the
section. A plot of the variation of $\frac{\theta_{tu}}{\theta_u}$ with $(q-q^1)/q_b$ for
beams having a variety of z/d ratios results in the following
equation.

$$(3) \quad \frac{\theta_{tu}}{\theta_u} = 1 + \left[(1.14\sqrt{\frac{\theta}{d}} - 1)\right] \times \left[1 - (\frac{q-q^1}{q_b})\right]\sqrt{\frac{d}{16.2}}$$

θ_{tu} = total inelastic rotation at ultimate, occurring
          between the section of maximum moment and an
          adjacent section of zero moment.

θ_u = inelastic rotation at ultimate, occurring within a
          length $\frac{d}{2}$ to one side of the section of maximum
          moment.

q   = tension reinforcement index

q^1 = compression reinforcement index

q_b = tension reinforcement for balanced ultimate strength conditions

This equation is the "best fit" straight line for the individual groups of data represented in this series of tests. In the equation q_b is calculated using $\varepsilon_u = .003$. This equation can be used in conjunction with the calculated value of θ_u to predict a reasonably safe limit for the value of the total inelastic rotation in the hinging region, θ_{tu}.

$$\text{Inelastic Rotation } \theta_u = \phi_u - \phi_y \frac{Mu}{My}$$

Rotation ϕ at ultimate $\phi_u = \overline{\tau}_u \frac{d}{2}$ ($\overline{\tau}_u$ = curvature at ultimate)

Rotation ϕ at commencement of yield of tension steel $\phi_y = \overline{\tau}_y \frac{d}{2}$

($\overline{\tau}_y$ = curvature at commencement of yield of tension steel)

Again it should be mentioned that the results in this investigation leading to the two preceding equations were for a limited testing study, and the equations are tentative pending further investigation.

PLASTIC HINGING AT A BEAM-COLUMN CONNECTION

An investigation was conducted by Ned H. Burns, A. M.
ASCE, and Chester P. Siess,[10] F. ASCE, to obtain a better under-
standing of the plastic hinging that develops at the connection
of a beam to a column in a frame. The study was limited to
the case with no axial load, and (to simulate a beam-column
connection) simple beams loaded through a stub were used. The
variables in the tests included tension steel ratio, compression
steel ratio, concrete strength, length of a column stub, rate
of loading, and depth of beam (Fig. 11). All beams were under-
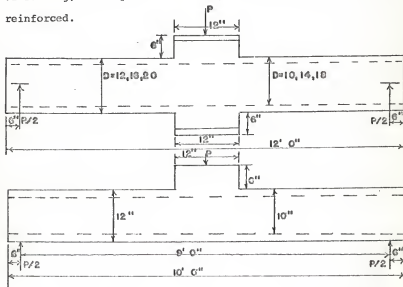reinforced.

FIG. II. EXAMPLE TEST SPECIMENS

A load-deflection curve was recorded, and the first break in the curve occurred when the beam cracked in the tension zone. This first flexural crack reduced the stiffness of the section, and changed the slope of the load-deflection curve accordingly. The second stage in the load-deflection curve was the yield point for the member. The tension steel yielded before the crushing strain was reached in the concrete. This yield point marked the boundary between elastic and inelastic behavior. Beyond the yield point there was a large increase in deflection for relatively little increase in load. As the load increased slightly a point was reached where first crushing of the concrete occurred, but the crushing of the concrete was so gradual that the exact point at which it first started to crush was very difficult to determine. The first visible notice of crushing was surface spalling on the top of the beam adjacent to the stub. The load continued to increase gradually beyond the point of initial crushing due to strain hardening of the tension steel. The maximum load was not reached until the crushing of the concrete had extended to the top of the stirrups and compression steel. At maximum load, the lateral deformation of the concrete increased and caused tensile strains in the closed stirrups. This forced the concrete core within the stirrups to behave similarly to a spiral column.

For beams without compression reinforcement, the internal compressive force was carried by the concrete core when the crushing became extensive. Failure occurred when the concrete core could no longer carry the compressive force. The presence

of compressive steel increases the ductility of the beam and also the ultimate load. Below ultimate load the compressive steel will act with the concrete in resisting compression until the steel yields. At ultimate load, the strain of the compression steel was well into the yield zone, but the compression steel does assist the stirrups in confining the concrete core which now has to take all the compressive force.

Another mode of failure involved buckling of the compression steel at ultimate load. When the compression steel buckled, the concrete core was forced to carry the transferred load, and resulted in crushing of the concrete and a decrease in the load as the buckling continued.

By increasing the amount of compression steel, the ultimate load could be increased. Failure occurred when the concrete crushed and the steel buckled, and this was also accompanied by a shearing moment along an inclined crack for deeper beams. In most cases failure occurred by initial crushing of the concrete followed by buckling of the compression steel.

The typical sequence of behavior for most tests was cracking, yielding of tension steel, crushing of concrete, and ultimate load capacity. It was found that the yield load increased with depth, giving smaller yield deflections for deeper beams. The strain hardening of the tension steel results in an increase in load after primary yield. For deeper beams there was more strain hardening due to the smaller steel percentage. The increase in beam depth also resulted in greater difference between yield and crushing load as well as between yield and ultimate load.

⌜The most interesting effect of the increase in compression steel is the resulting increase in ductility. The addition of compression steel has little affect on the deflection at crushing load. However, the added deformation capacity of the beam before failure is significantly increased. This is particularly useful for earthquake or blast loading. The deeper beams showed little increase in ductility.⌟

⌜The effect of tranverse reinforcement showed that closely spaced closed stirrups in the area where extensive crushing may be expected adds confinement to the concrete compression zone and makes the beam more ductile.⌟ The tests also showed that a closer spacing of stirrups in the area of highest curvature next to the column steel would be very effective in preventing the shear failure previously mentioned. The most effective use of stirrups would be to prevent shear failure and provide sufficient confinement to the concrete compression zone to force the beam to fail by fracture of the tension steel.

In those tests special emphasis was placed on the four loading stages, cracking, yielding, crushing, and ultimate. In order to assure this sequence of events, all the beams in the tests were under reinforced. It became apparent that two beams with the same properties, except for the omission of compression steel in one, had approximately the same load carrying capacity, but the effect on ductility was clearly evident.

It can be seen from these tests that one can assure almost any degree of ductility desired to permit redistribution of moment, if proper attention is given to details at points where first plastic hinging will occur in a structure.

CHAN'S STUDY OF ROTATIONAL CAPACITY

In ultimate load design of statically indeterminate structures, it is assumed that plastic hinges develop their full plastic moment at a particular point, while framework members between hinges remain elastic. It was the purpose of Chan's[11][12] tests to compare the idealized assumption of plastic hinges concentrated at points and the actual spread of plasticity.

Chan verified the relationship between θ values, as obtained from Baker's General Equation (which will be discussed later), and the actual development of plasticity in the hinge sections as established by Yu.

$$(4) \qquad \theta = \int^{\ell_p} \frac{m}{(EI)_p} \, dx - \int^{\ell_p} \frac{m}{(EI)_e} \, dx$$

ℓ_p = length of spread of plasticity along the longitudinal axis of the member.

m = moment at sections along yield length

$(EI)_p$ = EI Value after yield

$(EI)_e$ = EI Value before yield

From the preceding equation it is evident that the length of yield is a function of the moment-strain curve of the section and the shape of the bending-moment diagram due to external loads as shown in Fig. 12. If the length of yield, ℓ_p, is determined from the above relationship, then θ is represented by the shaded area of the curve as shown in Fig. 13.
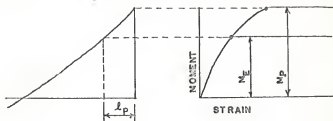
FIG.12. BENDING MOMENT & MOMENT STRAIN DIAGRAMS
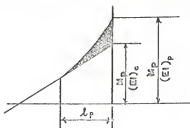
FIG.13. CURVATURE DIAGRAM

Chan also demonstrated that the ultimate strain of the concrete could be controlled by binders or stirrups placed at the hinge section. It has been shown that concrete strain can be safely increased to .01, and at this high level it would be possible to accommodate all practical modes of moment distribution in a redundant structure. The permissible strain may be limited though by cracks and deflection under working load.

Chan's analysis is based on the assumption that the
state of stress near the support of a beam or junction of a
column is similar to the case of simple bending. This is
very doubtful however, and a more complex distribution of
localized stresses is very likely to occur.

The idealized assumption of plastic hinges concen-
trated at points is satisfactory, and deformation of critical
sections and ultimate strength of critical sections in rein-
forced concrete frameworks may be determined. In continuous
beams with uniform loading the plastic length does not exceed
$\frac{L}{10}$; axial load in columns may increase this length to a maxi-
mum of $\frac{H}{2}$, but the assumption of point hinges is still valid.
Plastic rotation of under-reinforced sections may be increased
by using lateral binding. The binding will also increase the
stress-strain capacity. Finally, bound concrete can be used
to help eliminate brittle collapse normally associated with
compression failure.

LEE'S STUDY OF ROTATION CAPACITY

In 1955 L. H. Lee[7] suggested that, by assuming a
stress-strain curve in concrete compression, a relation-
ship between moment and curvature could be established
which could be used as follows:

By differentiating the general equilibrium equation,

$$(5) \quad fc = pE_s \ (e_c \frac{de}{de_c}s + 2 \ e_s \frac{de}{de_c}s \text{ and } e_s)$$

where fc = stress in concrete

e_s = tension strain in steel

e_c = compression strain in concrete corresponding
        to fc.

A curve for fc can be plotted with respect to e_c by measuring
e_c and e_s from beam tests. The stress-strain relationship
could be approximated by $fc = He_c - Be_c{}^2$, where H and B are
constants equal to 2fcm/ecm and fcm/(ecm)² respectively.
The maximum compressive stress in the concrete is denoted by
fcm, and ecm denotes its corresponding strain.

Letting X represent the curvature at a distance n from
the neutral axis, then the strain of a failure at that distance
is Xn, and the corresponding stress equals $Hxn - Bx^2n^2$. Using
equilibrium in horizontal forces the following equation is
obtained:

$$(6) \quad K^2 \ (\frac{H}{2} - \frac{BXKd}{3}) = pE_s \ (1 - K)$$

where K is the depth of the neutral axis. The resisting
moment expressed in terms of these parameters is:

$$(7) \quad M_r = bd^3 \ K^2 \ X \ \frac{H}{6} \ (3-K) - \frac{B}{12} \ X \ Kd \ (4-K)$$

A relationship between X and M can be derived by eliminating
K in the two preceding equations. Once this relationship is
known, distribution of moment due to plasticity can be deter-
mined by the conventional moment-area method.

THEORIES OF LIMIT DESIGN

For purposes of calculations an idealized moment-curvature relationship can be used to establish the load-moment relationships for any of the critical sections in a statically indeterminate concrete structure (Fig. 14).
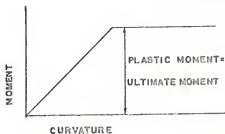
FIG. 14. IDEALIZED MOMENT- CURVATURE CURVE

The idealized moment-curvature curve implies that the hinging section will maintain its full plastic moment while developing enough rotation to allow full redistribution to occur. Therefore, full distribution of moment in a concrete structure depends upon the rotational capacity of the hinging sections. Because of this condition, it is necessary to develop a theory of limit design for concrete structures which is different from those already existing for steel structures.

The design theories of Professors G. C. Ernst, H. A. Sawyer, and A. L. L. Baker will be briefly presented as being representative of the many theories that have recently been developed.

DESIGN THEORY OF PROFESSOR G. C. ERNST

Professor G. C. Ernst used a unit rotation diagram to [13,14]
include the behavior of structures in the inelastic range.
The unit rotation at a section developing its ultimate
strength is $\phi_u = (E_u \& E_s)/d$, where d is the effective depth
and E_u and E_s are the maximum concrete and steel strains at ulti-
mate strength. Fig. 15a shows a beam with a finite yield length,
and Fig. 15b shows a beam with localized yield. In the second
case the yield has spread over a length λ, the the total concen-
trated rotation at a yielded section is $\theta_p = \lambda(\phi_u - \phi_0)$, where
ϕ_0 is the unit rotation at the commencement of steel yielding.
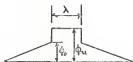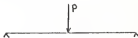
a)BEAM WITH FINITE YIELD LENGTH   **b)BEAM WITH LOCALIZED YIELD**

FIG. 15. UNIT ROTATION DIAGRAMS

Professor Ernst has applied the use of rotation diagrams
to the analysis of continuous reinforced concrete beams. In

this application the concentrated plastic rotations are considered to act as concentrated loads on the conjugate beam, and these rotations are checked at all hinging sections if full redistribution of moments is to occur. These calculated rotations are then compared with experimental values as to whether they are admissible.

DESIGN THEORY OF PROFESSOR H. A. SAWYER

Professor H. A. Sawyer[15] investigated the moment-rotation
characteristics of reinforced concrete beam sections, and has
used the observed characteristics to develop a theory of limit
design. Professor Sawyer used an idealized moment-curvature
diagram in which the moment in the plastic region increases
linearly from the moment at the beginning of steel yield (Me)
up to the ultimate moment of the section (Mu). From this
diagram the total plastic rotation can be calculated in sections
adjacent to points of maximum moment.
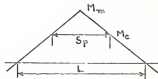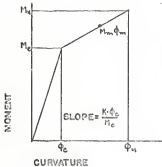
MOMENT DIAGRAM

$\theta_P =$ TOTAL PLASTIC ROTATION

$$= \frac{K \cdot \phi_c}{M_c} \left(\frac{M_m - M_c}{2} \right) S_p = \frac{K}{2} \cdot \frac{\phi_c}{M} \cdot \frac{(M_m - M_c)}{M_m} \cdot L$$

SLOPE $= \frac{K \cdot \phi_c}{M_c}$

MOMENT

CURVATURE

a) IDEALIZED MOMENT CURVATURE

b) TYPICAL CALCULATION OF TOTAL
PLASTIC ROTATION

FIG.18. LIMIT DESIGN FOR CONCRETE STRUCTURE

A trial and error adoption of the elastic center method of analysis is used to analyze continuous bays and single-bay portals. The total plastic rotation in any region of maximum moment is assumed to be concentrated at the point of maximum moment. A constant stiffness is assured for all points away from points of maximum moment. A trial and error process is used until the components of movement of the elastic center due to the effect of both elastic and plastic deformations is zero.

DESIGN THEORY OF PROFESSOR A. L. L. BAKER

Professor A. L. L. Baker[5] has developed the most widely
accepted theory of limit design. This theory postulates that
a structure, prior to failure, will develop a number of yielded
sections equal to the degree to which the structure is inde-
terminate. The yielding is concentrated at the yielded sec-
tions, and the members between the points of yield are assumed
to remain elastic. The ultimate strength of the section is
reached when a small additional load will create an additional
hinge, thus causing the structure to collapse.

When the number of yielded sections equals the degree to
which the structure is indeterminate, the structure becomes
statically determinate. The plastic rotations of the hinging
sections (θ_1, θ_2, etc.) can be calculated using the standard
influence coefficient equations as follows:

$$(8) \quad d_{11}\ \bar{x}_1 \ \& \ d_{12}\ \bar{x}_2 \ \& \ . \ . \ . \ . \ . \ + d_{1n}\ \bar{x}_n \ \& \ d_{10} = -\ \theta_1$$
$$d_{21}\ \bar{x}_1 \ \& \ d_{22}\ \bar{x}_2 \ \& \ . \ . \ . \ . \ . \ + d_{2n}\ \bar{x}_n \ \& \ d_{20} = -\ \theta_2$$
$$. \ .$$
$$. \ .$$
$$d_{n1}\ \bar{x}_1 \ \& \ d_{n2}\ \bar{x}_2 \ \& \ . \ . \ . \ . \ . \ \& \ d_{nn}\ \bar{x}_n \ \& \ d_{no} = -\ \theta_n$$

where,

$\bar{x}_1$, $\bar{x}_2$, etc. are plastic moments at each yielding section,
and d_{11}, d_{12}, $\longrightarrow$ d_{1n}, etc. are the influence coefficients for
the structure, (d_{mn} is the rotation of hinge m when a unit moment
is applied at hinge n),

$$(9) \quad d_{mn} = \frac{Mm Mn ds}{EI} \text{ around the structure,}$$

Mm = moment at any point in the structure when $\bar{x}_m = 1$
and all other $\bar{x}$'s are zero,

Mn = moment at any point in the structure when $\bar{x}_n = 1$
and all other $\bar{x}$'s are zero,

$d_{10}, \ d_{20} \longrightarrow d_{no}$, etc. are the rotations at hinges 1, 2 $\longrightarrow$ n,
due to the applied loads when $\bar{x}_1, \ \bar{x}_2, \ \longrightarrow \bar{x}_n$ are zero,

$$(10) \quad d_{no} = \frac{Mn \ Mo \ ds}{EI} \text{ around the structure,}$$

Mo = moment at any point in the structure due to applied
loads where $\bar{x}_1, \ \bar{x}_2, \ \longrightarrow \bar{x}_n$ are all zero.

The equations are applied as follows: Values are assigned
to the plastic moments $\bar{x}_1, \ \bar{x}_2,$ etc. and the plastic rotations
$\theta_1, \ \theta_2,$ etc. necessary for full redistribution are calculated
using the equations. The chosen values of $\bar{x}$ must give positive
θ values less than the permissible values for θ, otherwise the
chosen values of $\bar{x}$ must be adjusted until the values for θ are
acceptable. These fundamental equations are applicable to
all types of statically indeterminate concrete structures.

These equations may also be used to obtain an approximate
solution for the structure in the elastic state. For this
procedure $\bar{x}_1, \ \bar{x}_2,$ etc. must be adjusted until $\theta_1, \ \theta_2,$ etc. are
very nearly zero. Now the bending moment distribution may
be used to check the working load stresses.

The following expressions are used for limiting values of
rotation θ:

$$(11) \quad \theta = \frac{Ep1n}{K_u d} \text{ for sections in which tension occurs,}$$

(12) $\theta = \frac{(Ep-Es)}{d} \, |p$ for sections entirely in com-

pression,

where,

Ep = increase in maximum strain of concrete after
beginning of yield of steel,

Es = strain in reinforcement at least stressed face,

|p = length of yielding section

d = effection depth of section

$K_u d$ = depth of neutral axis at crushing of concrete.

Professor Baker recommended safe limiting values suitable
for design as follows: $E_u = 0.001$; $(E_u - E_s) = 0.001$; $|p = d$.

FOREIGN LIMIT DESIGN CODES

A brief discussion of foreign design codes is given to illustrate the practical use of limit design.

The British code[16] allows a 15% redistribution of moment at the supports, provided that these modified negative moments are used for the calculation of the corresponding moments in the span. The Norwegian code[17] allows a 25% reduction of support moment due to live load without correspondingly increasing the span moment. A comparison will show that redistribution of support moment for the British code is considerably greater, provided the ratio of live to dead load is small. For continuous beams the sum of the positive and negative moments is practically identical according to the two codes, and is independent of the dead to live load ratio. In this case, there is no difference in load carrying capacity of any two beams carrying the same load, if each is designed by a different code. It has been demonstrated that the 15% redistribution is more than safe, because it could have easily taken place under working load due to cracks in the concrete section, although the structure might have been designed by elastic theory.

The Danish code[18] recommends that the moment at a section should not be less than 1/3 of that obtained by the elastic theory. This means that a redistribution of 66 - 2/3% is allowed. The code recommends, and rightfully so, that limit design can only be used by those familiar with the theory, and then only with careful judgement and thorough analysis.

The Russian Commission for Scientific Research[12] recommended the following slab and beam formulas:

$\frac{W l^2}{16}$ for span

for indeterminate spans of slabs and beams

$\frac{W l^2}{16}$ for support

$\frac{W l^2}{11}$ for span

for end spans of slabs

$\frac{W l^2}{14}$ for support

$\frac{W l^2}{12}$ for span

for end span of beams

$\frac{W l^2}{12}$ for support

The moment arbitrarily chosen at any section should not be less than 70% of that obtained by elastic theory. The reinforcement should be such that the ratio of the depth of the neutral axis to the effective depth is not less than 0.3. These last two limitations will prevent excessive cracks and deflection in the working load stage.

CONCLUDING REMARKS

Limit design is a logical extension of ultimate strength design principles. It is a sound engineering design procedure, and will lead to better, more economical concrete structures. Limit design considers moment redistribution in calculating the maximum load capacity of an indeterminate structure, and also permits an intelligent arbitrary choice of redundant moments. Limit design would allow a reduction of negative support moment and avoid congestion of reinforcement, especially where negative beam reinforcement in two directions intersects the column reinforcement. Limit design permits a realistic evaluation of the effect of settlement and value change on load capacity. Finally, limit design permits control of flexural cracking which is particularly important for aesthetic reasons. The advantages of limit design mentioned above could be introduced into American design practice in either of two ways: First, by the introduction of a theory of limit design similar to that proposed by A. L. L. Baker; second, by introducing clauses in design codes which would permit arbitrary adjustment of design moments calculated using elastic theory. (These adjusted moments would have to stay within prescribed limits.)

Further analytical studies should be made of limit design, so that simplifications in design methods can be made for simple redundant structures. Next, the evaluation of hinge rotation for three dimensional frames should be developed.

Further analytical investigation of the effect of buckling at plastic hinges for a long slender member should be carried out, along with simplified solutions for cases of partial collapse and for elasto-plastic areas.

Previous tests have only begun to scratch the surface of the precision to which limit design can be refined. Further testing should be carried out to observe the following: the crack distribution and deflection at various stages of redistribution; the amount of moment redistribution as a function of reinforcement precentage; a more exact evaluation of hinge rotation; the spread of plasticity along the longitudinal axis of the member at the yielding zone; the effect of shear on hinge rotation; finally, the state of stress at the junction of a three-dimension framework.

The inelastic behavior of concrete structures at high loads has been demonstrated many times; with the wide acceptance of this characteristic of structural concrete, and the favorable results of past analytical and experimental work, it is reasonable and logical that the recognition of limit design be considered most carefully and expeditiously in order that design practice will reflect the true behavior of concrete structures.

ACKNOWLEDGMENT

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DEVELOPMENT OF APPLICATION OF PLASTIC HINGES TO
REINFORCED CONCRETE THEORY

by

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B.S., UNIVERSITY OF MASSACHUSETTS, 1966

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AN ABSTRACT OF A MASTER'S REPORT

Submitted in partial fulfillment of the
requirements for the degree

MASTER OF SCIENCE

Department of Civil Engineering
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Inelastic behavior of reinforced concrete has played an important part in structural design. The first step was the introduction of ultimate strength design, and the second was the introduction of limit design which allows readjustment in the relative magnitudes of internal forces and moments at sections subjected to high loads. It is the purpose of this report to present a review of the development of limit design of structural concrete, and stress the importance of incorporating limit design into United States building codes.

A brief section of the report is devoted to introducing the reader to the basic concepts of plastic design of steel frames and the limit design of structural concrete. These differences are the distribution of moment resistance and the rotation capacity.

A historical review of the experimental results of moment redistribution in reinforced concrete is presented, starting with the experimental work of the German Reinforced Concrete Committee through the work of L. H. N. Lee and A. L. L. Baker in the early 1950's. The results of this testing verify that moment redistribution does occur, but more extensive testing is necessary to predict the safe degree of redistribution in any one particular structure.

Various equations for predicting the rotational capacity of hinging regions are presented. It should be mentioned that most of the equations were derived for a limited testing study, and the equations are tentative pending further investigation.

As a result of the experimental investigations of moment redistribution and rotational capacity of reinforced concrete, various theories of limit design were introduced, the most widely accepted of these being that of Professor A. L. L. Baker. Professor Baker developed influence coefficient equations to calculate the plastic rotation of hinging sections, and recommended safe limiting values suitable for design.

A brief discussion of foreign design codes, British, Danish, and Russian, is given to illustrate the practical use of limit design. With the general acceptance of limit design by various foreign countries, and the favorable results of past analytical and experimental work, it is the conclusion of the writer that recognition of limit design be considered most carefully and expeditiously, in order that design practice will reflect the true behavior of concrete structures.